BACK PAPER 2019: COMPUTER SCIENCE II (B.MATH. 2ND YEAR)

- *This is a pen-and-paper, closed book exam. Use of computers/calculators is not allowed.*
- *All questions are compulsory. The question paper is of* 100 *points.*

Problem 1 ($2 \times 5 = 10$ marks)**.** Answer the following questions. You are not required to furnish an explanation or demonstrate the calculation.

(1) What does the command `print sec(1)` will output in `SageMath`?

(2) What does the following program output?

```
for i in range(10):
    print i
```

(3) What is the order of global truncation error needed to guarantee convergence and consistency for numerical solutions of ordinary differential equations?

(4) Given N data points (x_i, y_i) for $1 \leq i \leq N$ where $x_i \in \mathbb{R}^r$ is a r-tuple and $y_i \in \mathbb{R}$. Then we can find the best fit linear relation F for $y = F(x)$ using QR-decomposition of an appropriate matrix. There does not exist a best fit linear relation when $x_i \in \mathbb{C}^r, y_i \in \mathbb{C}$ since QR-decomposition exists only for real matrices. True or False?

(5) A Gaussian quadrature which computes weighted sum of n evaluations has zero error when computing the integral for a polynomial of degree d. What is the largest possible value of d?

Problem 2 (10 marks)**.** What does the following program output?

```
k = 0
for i in range(100):
    k = RealNumber(k + 2^(-i))
print k.str(base = 2)
```

Problem 3 (10 marks)**.** Consider a tridiagonal square matrix of real numbers, of size $N \times N$ − that is a matrix $A = (a_{ij})_{1 \leq i,j \leq N}$ where $a_{ij} \neq 0 \Rightarrow |i-j| \leq 1$. Assume that $|a_{ii}| > |a_{i,i-1}| + |a_{i,i+1}|$ for all $1 \leq i \leq N$ where we define $a_{1,0} = a_{N,N+1} := 0$. Show thatA^{-1} exists.

Problem 4 (10 marks)**.** The following `SageMath` function was supposed to produce the derivative of the piecewise polynomial function $p(x) := \begin{cases} x^2 - x + 1 & x < 1 \\ 2x^2 - 3x + 2 & x \geq 1 \end{cases}$ which is differentiable everywhere. It is flawed. What is it's output and why?

```
def p(x):
    if (x<1):
        return x*x-x+1
    else:
        return 2*x*x-3*x+2

print p(x).derivative() #Supposed to print the derivative!
```

Problem 5 (20 marks)**.** Let $f : [a,b] \to \mathbb{R}$ be a function which is infinitely differentiable in some neighborhood of $[a,b]$. Let $L_n(f)$ be the degree n Lagrange polynomial approximating f, that is interpolating $n+1$

points $\left(a_k := a + k\left(\frac{b-a}{n}\right), f(a_k)\right)$ for $0 \le k \le n$. Show that

$$|f(x) - L_n(f)(x)| \le \frac{(b-a)^{n+1}}{(n+1)!} \sup_{z\in[a,b]} |f^{(n+1)}(z)|.$$

Problem 6 (20 marks)**.** Compute QR decomposition of the following matrix

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix}$$

that is find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

Problem 7 (20 marks)**.** Assume $f : [a,b] \to \mathbb{R}$ is a function which is infinitely differentiable in some neighborhood of $[a,b]$. We compute $\int_a^b f(x)dx$ using Newton-Cotes integration rule of degree d (that is by using piecewise polynomials of degree d) and let us denote the result as $I_d(f)$.

(1) Let d be odd. Show that for $f(x)$ a polynomial of degree less than or equal to d, the method computes the integral exactly – that is $I_d(f) = \int_b^a f(x)dx$.
(2) Let d be even. Show that for $f(x)$ a polynomial of degree less than or equal to $d+1$, the method computes the integral exactly – that is $I_d(f) = \int_b^a f(x)dx$.

—THE END—